

The Hebrew Mathematical Tradition

Part One (Y. Tzvi Langermann)
Numerical Speculation and Number Theory. Geometry.

Part Two (Shai Simonson^{†*})
The Evolution of Algebra

1. Numerical Speculation and Number Theory. Geometry.

In this part, we look in detail at the arithmetic and numerology of Abraham Ibn Ezra. We continue with a broad description of Hebrew contributions in geometry. In part two, we shift the focus to algebra, and its evolution from a geometric to a combinatorial subject, as seen in the works of Levi ben Gershon and Abraham Ibn Ezra.

1.1 Numerical Speculation and Number Theory

Abraham Ibn Ezra (1092-1167) is the central figure in the Hebrew mathematical tradition. It was he who wrote what is probably the most influential Hebrew treatise on arithmetic, *Sefer ha-Mispar (Book of the Number)*; *inter alia*, that book seems to have played an important role in the transmission of the Hindu-Arabic numerals to the west. In addition, it was he who, by means of some enticing, cryptic remarks in his biblical commentaries, suggested a deep mathematical interpretation for some fundamental religious concepts. His commentaries were very widely read; they engendered a flock of supercommentaries whose authors did their best to spell out in full the ideas which Ibn Ezra only hinted at. Ibn Ezra also wrote *Sefer ha-Ehad (Book of the One)*, an arithmological¹ monograph, and *Sefer ha-Shem (Book of the Name)*, a reflection upon the divine names in which some mathematical themes are developed as well.

Ibn Ezra's most significant musings as found in a few excursions from his verse by verse explication of the Torah. Let us look at one of these, which is found in his remarks to Exodus 33:21. The properties of the tetragrammaton YHWH are discussed; and the numerical values of the letters of the Hebrew alphabet (Y=10; W=6; H=5) are the basis for Ibn Ezra's observations. We shall translate the text phrase by phrase, elaborating within square brackets upon the extremely terse wording; we readily acknowledge our debt to some supercommentaries written in the fourteenth century.

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¹Arithmology is a term coined, I believe, by Armand Delatte, one of the premier historians of Pythagoreanism; it refers to the class of writings dealing with the mathematical properties of the first ten numbers and their significance for religion and philosophy. Note that Ibn Ezra's monograph deals only with the first nine numbers.

“Said Abraham, the author. I have already commented upon the name that is written but not pronounced. It is a proper noun [*shem ha-esem*, literally, “the name of identity”], and its identity is the glory. When you add up all of the letters, it amounts to 72. [The sum here is: $Y+YH+YHW+YHWH = 10+15+21+26 = 72$.] This why our sages said that it is the ineffable name. [In fact, it seems that the notion of a name composed of 72 letters is post-Talmudic.] If you add the square of the first to the square of the middle, in truth it is like the number of the Name. [The squares on the first number, 1, and on the middle of the first ten digits, i.e. 5, add up to 26, which is also the sum of the numerical values of YHWH.] Likewise the conjunctions of the five planets. [There are 26 possible combinations of the five planets, taken five, four, three, or two at a time.] So also, if you add up the letters which a person pronounces in half of the Name, they are equal to the number of the Name. [If one pronounces the first letters, *yod*—written YWD, and numerically equivalent to 20, and *he*, written HA and numerically equivalent to 6, then they too add up to 26.] If you take the sum of the squares of the even numbers of the first grade, it is equal to the collective sum of the half of the name. [$2^2 + 4^2 + 6^2 + 8^2 = 120 = 1+2+3+ \dots +15$; 15 is the value of YH, which is half of the Name.] If you multiply half of the Name by the other half, the result is the [sum of the] squares of the odd numbers [of the first grade]. [$YH \cdot WH = 15 \cdot 11 = 165 = 3^2 + 5^2 + 7^2 + 9^2$.] If you subtract the square on the first from the square on the [first] two, the result is the cube on the second letter. [The square on the first letter, $Y=10$, is 100; and the square on the first two letters, $YH=15$, is 225; their difference is 125, the cube on the second letter, $H=5$.] And if you subtract the square on the first two from the square on the first three, the result is the cube on the third letter. [The square on the first three letters, $YHW=21$, is 441; and $441-225=216=6^3$.] This Glorious Name is the One which is self-standing. It does not require anything before it. For if you look at it from the aspect of counting, where it is the beginning of the All, and all counting derives from ones, [then] it is the one which is the All.”

It is important to define as precisely as we can the type of “number mysticism” which Ibn Ezra practices. Clearly, he is drawing upon the neopythagorean tradition, in particular, the work of Nicomachus of Gerasa (second century), whose widely repercussive *Introduction to Arithmetic* devotes considerable attention to series and their sums. As we have just seen, this is just the type of mathematical insight which appeals to Ibn Ezra. Moreover, “the All” is an important neoplatonic concept; and neoplatonic cosmology often went hand in hand with neopythagorean numerical speculations.

In particular, it is crucial to distinguish between Ibn Ezra's enterprise and the very popular hermeneutical exercise known as gematria. The latter find deep secrets in the numerical values of Hebrew words. Ibn Ezra unequivocally distances himself from the practice, for example, in his explication of Genesis 14:14, which narrates Abraham's mobilizing his 318 interns in order to rescue his nephew Lot. In his commentary to the verse, Ibn Ezra takes the bold step of rejecting a Talmudic homily (*Nedarim* 32a) which asserts that the 318 interns refer to just one person, Abraham's faithful slave Eliezer, since, in gematria, $ALY'ZR=318$. He states emphatically, “Scripture does not speak in gematria.”

According to the practitioners of gematria, secrets are contained in the words and letters of the Hebrew alphabet; arithmetic is a tool for ferreting out these secrets. In Ibn Ezra's view, the matter is reversed: it is number which contains the secrets, and the Hebrew spellings reflect the deep meaning which these numbers have. Thus, for example, the divine names YHWH and AHYH (pronounced *eheyeh*, it is the "I am that I am" of Exodus 3:14) are spelled with the letters which stand for the numbers 1, 5, 6, and 10, precisely because these numbers have tremendous mathematical import. As Ibn Ezra explains in the third chapter of his *Sefer ha-Shem*, the number one is "the cause of all number, but is not itself a number." Ten resembles one, since it is both the end of the units and the beginning of the "tens". Moreover, 5, 6, and 10 all share this property in common: all powers of these three numbers will maintain in the final digit the same final digit which they have in the first power. (So, e.g. $5^2=25$, $5^3=125$, and so forth). As it seems to me, the key concept here is cyclicity or replication, an idea which resonates well with the Pythagorean concept that numbers are the building blocks of creation. The other properties of these letters, their shapes, and their application in Hebrew grammar, all ramify from their peculiar mathematical characteristics in the base ten system.

Ibn Ezra was interested primarily in numbers rather than shapes; he specialized in arithmetic, not geometry. He was expert in astronomy and, accordingly, he was well-versed in geometry, as every astronomer must be. He seems to have regarded the circle as a sacred form. The Hebrew letter *yod*, which stands for the number ten, has the shape of a semi-circle, "to indicate the whole [circle]. The reason is, that it encircles the All" (*Sefer ha-Shem*, ch. 3). In his shorter commentary to Exodus 26:37, he depicts the candelabrum in the Tabernacle as semicircular; this unusual explication must also be traced to the meaning which he finds in that geometric form. One passage which Ibn Ezra included both in his commentary to the Torah (*ad* Exodus 3:14) and, with some elaboration, in his *Sefer ha-Shem* (chapter 6), Ibn Ezra tied together the special properties of the circle and the number of ten. As usual, his remarks are laconic and cryptic, which amplified the interest which they generated. Isaac Israeli, the eminent fourteenth-century astronomer, and Mordecai Comteano, the fifteenth century Byzantine polymath, both spelled out the theorem which Ibn Ezra only hinted at. With the aid of simple modern algebra, the purpose of Ibn Ezra's construction is readily perceived.

The passage reads as follows (in the commentary to Exodus): "If make the diameter of the circle equal to this number [ten], and draw a chord in its third, then the number of the isosceles triangle will be equal to the number of the perimeter, and so also the rectangle within the circle." In Figure 1, the diameter AB is assigned the number 10. Chord CD cuts off one-third of the circle, as does chord EF. Ibn Ezra asserts that the area or "number" of triangle ACD = rectangle EFDC. This is easily proven. Right triangles ECI and JIA are equal, as are right triangles FDK and JKA. The trapezium CIKD is common to triangle ACD and rectangle EFDC. Adding to the trapezium triangles ECI and FDK produces the rectangle, and adding their equal, i.e. triangles JIA and JKA produces the triangle ACD. Hence triangle ACD = rectangle EFDC.

This "number", in turn, is said to be equal to the perimeter of the circle, provided, we recall, that the length of the diameter has been given as 10. To be sure, the "numbers"

of the triangle and rectangle are in square units, whereas the perimeter is a linear measure; but Ibn Ezra is interested only in the number, not in what it measures. Now if we call the diameter d , then $GH = d/6$ and $GD = d/2$. Applying the Pythagorean theorem, we find that $HD = (d\sqrt{2})/3$, so $CD = (2d\sqrt{2})/3$, and the area of rectangle $EFDC = (2d^2\sqrt{2})/9$. The perimeter is πd . Since, as emphasized above, we are interested only in the “numbers”, we set the perimeter equal to the area of the rectangle; after dividing through by d , we are left with $\pi = (2d\sqrt{2})/9$. Simple computation (on a pocket calculator) shows that $(2\sqrt{2})/9$ is approximately equal to 0.314. Hence, only when $d = 10$ will the rectangle yield a fairly good approximation to π . This, we believe, is the property which Ibn Ezra wished to demonstrate.

Like other pre-modern mathematicians, Ibn Ezra regarded neither zero nor one to be numbers. Zero serves only as a place-marker in decimal notation; and one is the primal source from which numbers are generated. Thus the first true number is two. As he states in his *Sefer ha-Ehad*, “two is the beginning of all number.” Nonetheless, here and there in his mathematical writings, Ibn Ezra treats both zero and one just as he would any other number. We have already seen that he groups together the zero in 10 along with 5 and 6 when noting that all three are preserved in successive powers of their respective numbers. As for one: among the first properties listed in *Sefer ha-Ehad* are the facts that one is “square root and cube root, square and cube.” However, at the beginning of the seventh chapter of his *Sefer ha-Mispar*, he offers a prosaic explanation for one being equal to its own square, namely, that the squares of fractions are less than their roots, and the squares of whole numbers are greater. On the other hand, it is noteworthy that Ibn Ezra—in contravention of the arithmological tradition—does not include a chapter on the number ten in his *Sefer ha-Ehad*. True, *Sefer ha-Ehad* is concerned almost exclusively with mathematics and, as Ibn Ezra states at the end of the chapter of the one, “computation with the ten resembles that of the one”. In other words, with the number ten, we have simply moved into the next grade or power of numbers. Nonetheless, astrological, cosmic, and other meanings are adduced in *Sefer ha-Ehad* (especially in connection with the number 4), and moreover, the arithmological tradition—and Ibn Ezra's own religious musings, as we have seen above—attaches considerable significance to the number 10. Ibn Ezra's decision to omit this number from his arithmology demonstrates, at the very least, that he took the conformity of the one to the ten very seriously indeed.

1.2 Geometry

The branch of mathematics in which Jews applied themselves most diligently was geometry. There are a number of possible explanations why this was so. First, geometry was intimately connected with logic, especially logical demonstration. In the medieval period, algebraic rules were often merely illustrated by way of numerical example; but geometrical propositions—including those that are the equivalent of such basic algebraic formulae as the binomial expansion—were proven rigorously. Indeed, geometrical demonstration in some ways represented to thinkers of the period the pinnacle of scientific certainty, and ideal which other branches of learning could strive to emulate but, in general, not attain.

Closely connected to this is the fact that many of the basic concepts of medieval natural philosophy, including some notable notions which were of crucial significance for theology, were debated within a geometrical context. The very intricate (and as yet unstudied) discussions in Levi ben Gershon's commentaries on Averroes' *Physics* (who, in turn, is commenting upon Aristotle) are part of this tradition. Theologians generally agreed that the deity was the infinite, unbounded, noncorporeal mover of a finite and corporeal universe. Notions such as infinity, motion, and even corporeality were analysed geometrically. Hebrew theological tracts very often contain geometrical proofs connected to some of these notions.

Finally, we should take note of the close connection between geometry and astronomy. The Hebrew tradition in astronomy was very strong; and astronomers used geometrical models to describe the planetary motions. A sound education in geometry was requisite for any serious study of astronomy.

The corpus of Hebrew geometrical writings consists almost exclusively of translations. At first this may appear a bit strange: why are there not a few works on arithmetic, and even some on algebra, but almost none in geometry? The simple answer is that Hebrew scholars, like those throughout West and a good part of the East, relied upon the great synthesis of Euclid. The *Elements* were translated from the Arabic into Hebrew, along with the commentaries of al-Farabi and Ibn al-Haytham, and a number of glosses which explained differences between the two Arabic translations (by Thabit and al-Hajjaj). A smaller number of Arabic writings connected to the *Elements* (including the Arabic translation of the *Elements* itself) were transcribed into the Hebrew alphabet but not translated. Euclid's short optical treatises were also available in Hebrew, as were some writings of Archimedes and Eutocius.

There exist as well a few precious codices containing unique Hebrew translations of geometrical tracts—in some cases, the only surviving copy of a thread in geometrical thinking which occupied Hellenistic and Arabic as well as Jewish mathematicians. These manuscripts are a blessing for historians of mathematics; and they also tell us something about the cultural priorities of the medieval Jewish communities. The most important codex of this type is ms Oxford-Bodley d. 4 (Neubauer and Cowley 2773). Much like a modern university professor, worried that one shoddy piece of work could destroy a reputation established by years of hard work, the translator, Qalonymos ben Qalonymos (d. after 1329) appended a personal note begging the indulgence of his readers, and informing them that he was forced to work quickly and from a defective manuscript, so eager was his patron to obtain a Hebrew copy. Mss. Hamburg, Levi 113, and Milan, Ambrosiana 97/1, are two other noteworthy codices.

A few distinct topics generated particular interest, and around them there evolved a distinct Hebrew tradition. Perhaps the most important of these is the corpus of demonstrations of the property of an asymptote to a curve. Maimonides, in his *Guide of the Perplexed*, part 1, chapter 73, noted the asymptotes to the hyperbola, whose properties are demonstrated by Apollonius in his *Conics*. Maimonides studied the *Conics* in Arabic. However, that text was never translated into Hebrew, and Hebrew readers of the *Guide* required an independent demonstration of this particular property. In fact, the notion that

two lines could continuously approach each other without ever meeting stimulated thinkers in other cultures as well. In addition to Apollonius' discussion (for those with access to this *Conics* and the necessary background in order to understand it), Nichomedes' proof utilizing the conchoid was studied, and some original proofs based on Euclid were expounded. However, it seems that, in the wake of Maimonides' statement, Jewish mathematicians applied themselves to this problem with particular vigor. At least six distinct proofs are presented in the Hebrew literature (Lévy, 1989). Although the passage from the *Guide* was the immediate stimulus, interest in this issue finds its place in larger movements in cultural history; these have been explored by Freudenthal (1988).

Another interesting and hitherto unexplored Hebrew tradition is comprised by the texts on spherical geometry (often called sphaerics for short). Mastery of this branch of geometry was a necessary prerequisite for serious students of astronomy. However, the codicological evidence—by which I mean the existence of numerous codices exhibiting similar features and transmitted independently of astronomical writings—suggests that there was considerable interest in sphaerics from the point of view of pure mathematics. At the core of this tradition are two Hellenistic texts, treatises on the sphere written by Menelaos and Theodosios. The Hebrew versions were—as was the case nearly always²—translations from the Arabic. Short tracts by Jabir ibn Aflah and Thabit ibn Qurra on transversals are often included. Occasionally some other relevant materials are included; for example, the Milan codex cited above contains some lengthy and unstudied commentaries.

What gives this tradition its coherence and uniqueness are the extensive glosses that are found in the margins of most manuscripts. Most of these are initialed; the signatures have not been deciphered. Many are signed by the Hebrew letters *dalet* and *tav*, which may mean *divrei Thabit* (“the words of Thabit”) but also *divrei talmid* (“the words of a student”).

Hebrew scholars were not on the whole as excited by the quadrature of the circle as were their Latin counterparts. There exists but a single Hebrew monograph on the subject, written by a certain “Alfonso” who, recent scholarship suggests, is none other than the fourteenth century apostate and philosopher, Abner of Burgos. Unfortunately, his treatise, *Meyashsher ‘Aqov* (“Straightening out the Crooked”) survives in only one, incomplete copy. Alfonso clearly realized that squaring the circle is impossible within the framework of conventional, Euclidean geometry; he set as his goal the description of some sort of “higher reality” where this would be possible. The surviving fragments establish the philosophical and mathematical foundations upon which, presumably, this “higher reality” is constructed. The subtlety and erudition displayed in this treatise are in a class by themselves, and it is a pity that the complete treatise is not extant.

By way of Aristotle, Jewish scholars were acquainted with one of the earliest Greek attempts at a solution, namely Hippocrates' quadrature of the lunule (a crescent shaped figure constructed from two circles). According to Aristotle, Hippocrates thought that his

²There are a few instances of direct Hebrew translations from the Greek, for example, a passage from Pappus (Langermann, 1996, p. 52).

successful squaring of the lunule indicated that the entire circle could be squared; and the Stagirite made use of this as an example of faulty reasoning. About half a dozen Hebrew manuscripts contain various reconstructions of Hippocrates' proof, usually as glosses to Aristotle or his commentators.

2. The Evolution of Algebra

We present an in depth look at Hebrew medieval algebra by examining some of the work of Rabbi Levi ben Gershon (1288-1344) and Rabbi Abraham ben Meir ibn Ezra (1090-1167).

When we say algebra, we mean the study of solutions of equations, and polynomial identities. In ancient times, this was intrinsically tied up with geometric notions. The connection of algebra with geometry is prominent in Greek mathematics, and continues in the geometric proofs of the algebraic results of Islamic mathematicians. Eventually, this dependence disappeared, and the full power of algebra came to bear with symbolic notation and combinatorial identities.

When one looks at the history of algebra, by concentrating on results, it seems to have made little progress for over 3000 years, and then a sudden leap. After all, the solutions to linear and quadratic equations were well known to the Babylonians, and the solution to the general cubic equation did not come until the Renaissance. Throughout these 3000 years, the Greeks, Indians, Chinese, Moslems, Hebrews and Christians, seem to be doing no more than presenting their own version of solutions to linear and quadratic equations, which were well known to the Babylonians. Diophantus, Bhaskara, Jia Xian, al-Khwarizmi, Levi ben Gershon and Leonardo of Pisa, all provide examples of this.

However, the history of algebra is of course more subtle than that, with small changes accumulating slowly. There were many concepts which needed to mature before the breakthrough in algebra was ready to occur. Independence from geometry, a comfortable symbolic notation, a library of polynomial identities, and the tool of proof by induction, were all stepping stones in the history of algebra. Hence what at first seems like new presentations of the same old stuff, needs to be considered more carefully.

To this end, to appreciate this particular view, it is useful to focus upon the liberation of algebra from its geometric roots, and its subsequent connection with combinatorics. Our intention here is to present the algebra of Levi ben Gershon and Abraham ben Meir ibn Ezra, and to argue that although their results were perhaps already well known, their approach represents an evolution of algebra from a geometric subject to a combinatorial subject.

2.1 Levi ben Gershon and *Masseh Hoshev*

Levi ben Gershon (1288-1344), rabbi, philosopher, astronomer, scientist, biblical commentator and mathematician, was born in Provence (South France) and lived there all his life. Through his writings, he distinguished himself as one of the great medieval

scientists and a major philosopher. He wrote more than a dozen books of commentary on the Old Testament, a major philosophical work *MilHamot Adonai* (Wars of God), a book on logic, four treatises on mathematics, and a variety of other scientific and philosophical commentaries. *MilHamot Adonai* has a section on trigonometry and a long section on astronomy, including the invention of the Jacob's Staff, a device to measure angles between heavenly bodies used for centuries by European sailors for navigation, a discussion of the *camera obscura*, and original theories on the motion of the moon and planets. A complete bibliography on Levi and his work can be found in [Ke92].

Levi was highly regarded by the Christian community as a scientist and mathematician. He is referred to by them as Leo Hebraus, Leo de Balneolis or Maestro Leon. However, despite his originality and reputation, much of Levi's scientific and mathematical work did not heavily influence his successors. It is not clear to what extent he played a role in the transmission of Hellenistic mathematics from the Arab world to western Europe.

Levi's Mathematics

Levi's mathematics comprises four major works.

1. *Maaseh Hoshev*, Levi's first and largest mathematics book is extant in two editions, the first completed in 1321, and the second in 1322 [Si00a]. It is known best for its early illustrations of proofs by mathematical induction [Ra70].
2. A commentary on Euclid was completed in the early 1320's, with ready access to Hebrew translations. Included is an attempt to prove the fifth postulate.
3. *De Sinibus, Chordis et Arcubus*, a treatise on trigonometry in *MilHamot Adonai*, was completed in 1342 and dedicated to the Pope.
4. *De Numeris Harmonicis*, completed in 1343, was commissioned by Phillip de Vitry, Bishop of Meaux, and immediately translated from Hebrew into Latin. Philip was a musicologist interested in numbers of the form $2^n 3^m$ called harmonic numbers. Levi proves that the only pairs of harmonic numbers that differ by one are (1, 2), (2, 3), (3, 4) and (8, 9). The book is relatively short and the original Hebrew is lost.

Maaseh Hoshev, The Art of Calculation

Maaseh Hoshev, The Art of Calculation, is Levi's first book. The title is a play on words for theory and practice, *Maaseh* corresponding to practice and *Hoshev* corresponding to theory. Levi writes in his introduction to the book:

"It is only with great difficulty that one can master the art of calculation, without knowledge of the underlying theory. However, with the knowledge of the underlying theory, mastery is easy... and since this book deals with the practice and the theory, we call it Maaseh Hoshev".

Maaseh Hoshev is a major work in two parts. Part one is a collection of 68 theorems and proofs in Euclidean style about arithmetic, algebra and combinatorics. Part two contains algorithms for calculation and is subdivided into six sections:

- a. Addition and Subtraction
- b. Multiplication
- c. Sums
- d. Combinatorics
- e. Division, Square Roots, Cube Roots.
- f. Ratios and Proportions.

A large collection of problems appears at the end of Section (f) in part two. Not including this collection of problems, parts one and two are about the same size, and the problems are about a third the size of each part. The text in part two and the problem section refers often back to the theorems of part one. Levi lists Euclid's arithmetic, books 7-9, as prerequisite reading.

2.1.1 Square Root Extraction Algorithm from Levi ben Gershon's *Maaseh Hoshev*

Before we look at Levi's method, let's review the ancient Babylonian method which is intuitive and serves as a starting point in order to more easily understand Levi's method. This method may be familiar to many readers. Given N , it works by guessing some number as a first approximation of the square root of N , and then improving the guess repeatedly and converging to the correct answer.

Babylonian Method for Calculating the Square Root of N

Let a_i be the i -th approximation and let $a_1 = 1$ (any number is just as good). Then $a_{n+1} = (a_n + N/a_n)/2$.

For example, the first three approximations for $N=2$ are:

$$a_1 = 1$$

$$a_2 = 1.5$$

$$a_3 = 1.416666$$

The History and Ideas Behind the Babylonian Method

The Babylonian method is from 1800 B.C.E. The Babylonians as usual, left no proof or explanation of the idea. The method can be thought of both algebraically and geometrically.

Nowadays, perhaps due to all the drilling students have in algebra, the idea is often presented with a common sense algebraic approach. Algebraically, if you guess that the square root of N is Old , and Old is smaller than the actual square root, then N/Old will be bigger than the actual square root. Then their average, $(Old + N/Old)/2$ is a better guess than Old . This is a simple and straightforward explanation, but with a geometric view, numbers are distances and products are areas.

Geometrically, constructing the square root of N can be done by constructing a square whose area is equal to N . This can be accomplished by constructing a rectangle with area N , and repeatedly reconstructing the rectangle until it is close to a square. We start by setting the smaller side of the rectangle to Old , and the larger side to N/Old . Now we take half the excess of N/Old over Old and move it from the bigger side to the smaller side, thereby “squaring” the rectangle, please see the figure below. The new square has side $(N+N/Old)/2$. However, the area of this square is larger than the area of the rectangle by the amount of the small missing square in the bottom right corner of the figure. Hence our new guess is smaller than the actual square root, but larger than our old guess. Our new guess becomes the side of a new rectangle, and we repeat the process. As we do, the small missing square gets smaller and smaller, and hence the side of the squared rectangle gets closer and closer to the actual square root.

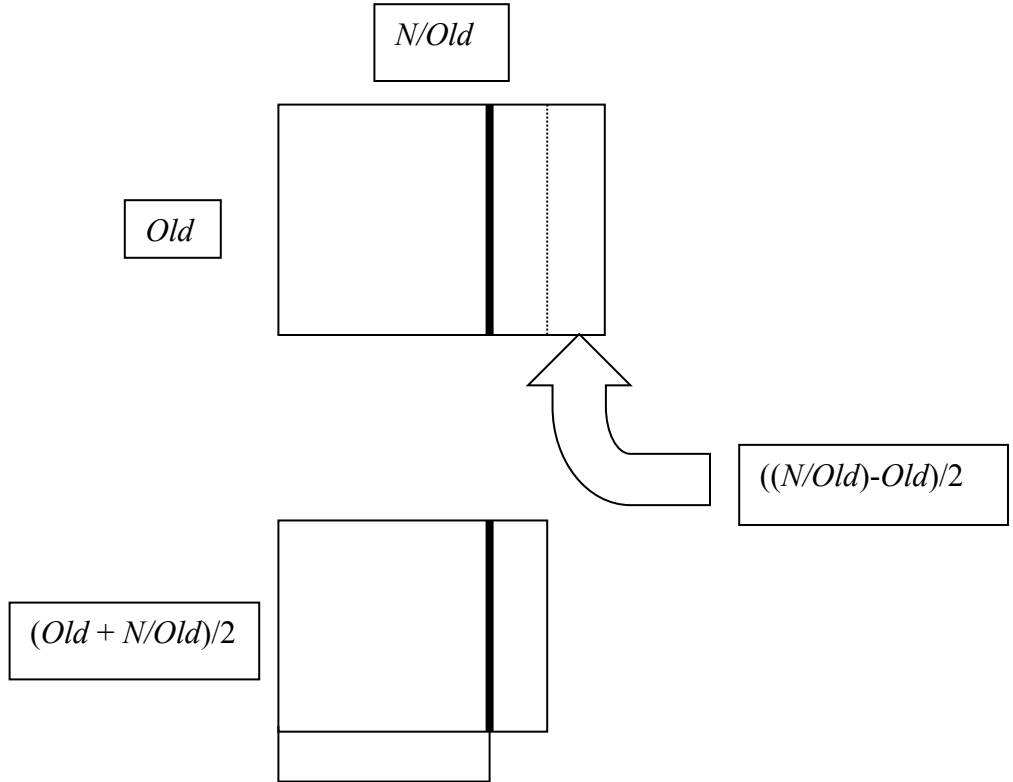


Figure 2. Geometric View of the Babylonian Method for Square Root Extraction

Levi's Method for Square Root Extraction

Levi's idea is not new. Some readers may recognize it as the method taught to them in school. It is seen in geometric form in Chinese sources that predate him by a few centuries. Indeed, it is natural to think of the algorithm geometrically, but Levi does not. He provides no figures, or references to any relevant geometric notion. It is noteworthy that his focus is simply on the algebraic identity $(a+b)^2 = a^2 + b^2 + 2ab$.

Levi's method for extracting square roots of perfect squares from Part 2 Section (e) of *Maaseh Hoshev* is shown below. An explanation of the algorithm in plain English appears afterward with some hints and analysis to help the reader unravel Levi's style. Be forewarned that Levi's writing is difficult. He makes no use of equations, and very little symbolic notation. Even the simplest relationships and theorems are stated in longwinded prose. It may be best to read this section lightly and return later for a more careful reading.

In order to extract the square root of a perfect square, write the number in a row according to its levels, and examine the last level to see whether it is odd. If it isn't odd, then bring down the digit from the level before it, so that the last digit is from an odd level.

Afterward, find the square closest to this number, but still smaller than it, and write the root of this square down in the root row, beneath the previous row, in the middle level between the first and last levels. And this is what we call the "result" row. And the square of this root is subtracted from the top row, and what remains is divided by twice the root. But be careful that after the division, there still remains an amount about as big as the square of the root that resulted from the division. Then write this result in the root row, in the level whose distance from the level that you divided is equal to the distance from that level to the first level.

Now take this result and multiply it by twice the root that exists, and by itself. This result is subtracted from the top row. And so you should do, until there is not one thing left in the top row.

For example, if you wished to find the square root of 82646281, since the last level is the eighth, bring down the one before it to give 82. And 81 is the square closest to this, and its root is 9. Write 9 down in the root row in the fourth level which is half way between the seventh and the first. And the square of 9 in the fourth, is 81 in the seventh. Subtract this from 82, and 1 in the seventh is left over. And this cannot be divided by twice 9, which is the current root, so bring down the one before it to give 16, and still we cannot divide by twice 9, so bring down the one before it to get 164. Divide this by twice the current root, which is 18, and the result is 9 which is the resulting root. And write this in the root row in the fourth level after the level of 164. Multiply it by twice the current root and itself, and subtract the result from the top row, and 18181 is left. Divide this by twice the current root, which on the previous side is 18 and 2 tenths, and the result is approximately 1 in the first level. And we write this in the root row in the first level. Multiply by twice the current root and itself, and subtract the result from

the top row, and there is nothing left in the top row. And so the square root of the given number is 9 thousand and 91, and that is what was requested.

And if you wish, you can check this by multiplying the root row by itself and getting the top row. And this works because of what I have already explained, that when one number is added to another, the square of their sum is equal to the squares of the two numbers and twice the product of one with the other.

			1	8	1	8	1
	1	6	4	6	2	8	1
8	2	6	4	6	2	8	1
				9	0	9	1

For those readers trying to untangle Levi’s presentation, there are a few things to note:

1. Levi is not completely consistent with his reference to the “top row”, nor does his example show a situation, where one must “be careful that after the division, there still remains an amount about as big as the square of the root that resulted from the division”, but there is still enough detail here, to extract the algorithm.
2. When Levi speaks of writing a digit in a particular “level”, he is referring to the column for that number. For example, 6 in the 4th level is his way of saying 6000.

An Explanation of Levi’s Method

Levi’s idea is to construct the square root one digit at a time, always making sure that the square of the current approximation remains less than the original number N . If the current approximation is a , then we need to account for $N-a^2$ by adding an appropriate b to the approximation. To this end, he reminds us that $(a+b)^2 = a^2 + b^2 + 2ab$, hence adding b accounts for $2ab+b^2$ of the required $N-a^2$. Given N and a , Levi computes b by dividing $N-a^2$ by $2a$. This gives a good guess, since $2ab+b^2 \approx 2ab$, when b is small in comparison to a .

Here is an example in his style. Let’s try 3224990521. Here the last level is the tenth level, so we look at 32 rather than 3. The largest square smaller than 32 is 25, and its root is 5, so we write 5 in the middle column (under the second 9), i.e. 50,000. Then subtract 25 from 32 and you get 7 which is not divisible by $2*5$, so we look at 72, which is divisible by $2*5$, and the result is about 7 in the fourth column, i.e. 7000. So we take 7 and multiply by $2*50 + 7$, giving us 749 which is bigger than 724. Whoops! We forgot that one must “be careful that after the division, there still remains an amount about as big as the square of the root that resulted from the division”. In this case, the square of the root is 49 and there is only 24 remaining after the division i.e. $724-700$. So we take 6 and multiply by $2*50 + 6$. This gives 636 which we can subtract from 724 giving 88. So our current root is 56,000. Now bring down another digit and divide 889 by $2*56$, and we get

approximately 7 in the third column (i.e. 700), which we multiply by $2 \cdot 560 + 7$ to get 7889, which we subtract from 8899 to get 1010. The current root now is 56,700. Bring down another digit giving 10100, so that we can divide by $2 \cdot 567$, and we get approximately 8 in the second column (i.e. 80). Multiply 8 by $2 \cdot 5670 + 8$ to get 90,784. Note that the 10100 is the prefix of 10,100,521, while 90,784 is the prefix of 9,078,400. So we subtract 90784 from 101005 to get 10221. The current root is now 56,780. Then bring down another digit to get 102212 so that we can divide by $2 \cdot 5678$, and we get approximately 9 in the first column (i.e. 9). Then multiply 9 by $2 \cdot 56780 + 9$ to get 1022121, and we subtract this from 1022121 which leaves us with nothing in the top row. So our final answer is 56789.

The following modern presentation of Levi's method is easier to follow but is not faithful to Levi's style or language.

Levi's Algorithm to Calculate the Square Root of a Perfect Square N

1. Pair up the numbers from right to left, and count the pairs. This leaves either a pair of digits at the left end or a single digit. Let P be the number of pairs, not counting the last digit or last pair of digits, and let M be the (one or two digit) number which remains on the left end. (For example, given 2415732, $P = 3$ and $M = 2$; given 2389397496, $P = 4$ and $M = 23$.)
2. Find the largest perfect square smaller than M , and let its square root followed by P 0's be the first approximation. Let A be this approximation.
3. Let $R = N - A^2$.
4. Divide R by $2A$, and let B be the result excluding all but the most significant digit. (Be very careful though, that $2AB + B^2$ is not larger than R . If so, then subtract one from the most significant digit of B and try again).
5. Let $R = R - (2AB + B^2)$. Let $A = A + B$.
6. If $R = 0$ then the answer is A and we are done, otherwise go back to step 4.

Here is a final example. Let $N = 16384$.

1. $P = 2$ and $M = 1$.
2. $A = 100$.
3. $R = 16384 - 10000 = 6384$.
4. $B = 6384/200 \approx 30$, but $2(100)(30) + (30)(30)$ is larger than 6384, so $B = 20$.
5. $R = 6384 - 2(100)(20) + (20)(20) = 1984$ and $A = 100 + 20 = 120$.
6. R is not 0 so go back to step 4.
4. $B = 1984/240 \approx 8$, so $B = 8$.
5. $R = 1984 - 2(120)(8) + (8)(8) = 0$, and $A = 120 + 8 = 128$.
6. $R = 0$ so the answer is 128.

Comparison of Levi's Method with the Babylonian Method

Levi's method, at first glance, seems very different from the Babylonian method. The main differences are:

- a. Levi always keeps his approximations smaller than the actual square roots and improves the approximations by adding small amounts, while the Babylonian method (except perhaps for the first guess) keeps the approximations greater than the actual square root, subtracting small amounts at each iteration.
- b. Levi's method converges from below at a steady rate of one digit per iteration, while the Babylonian method converges from above at an accelerating rate.

For example: Let $N = 16,384$. Then Levi's approximations will be 100, then 120, and finally 128. The Babylonian method will have the following approximations: 1, 8192.5, 4097.2499, 2050.6243, 1029.307, 522.6123, 276.9813, 168.0667, 132.7759, 128.0859, etc. It is interesting to compare the two methods with respect to ease of calculation and speed of convergence.

However, the similarity between Levi's method and the Babylonian method can be seen with some analysis and careful notation. If A is the current approximation of the square root, then the new approximation according to the Babylonian method is:

$$(A + N/A)/2 = A + (N - A^2)/2A,$$

and according to Levi it is:

$$A + \{ (N - A^2)/2A \},$$

(where the symbols \lfloor and $\{$ mean that the enclosed division excludes all but the most significant digit, and that this digit is occasionally decreased when necessary, as described in step 4 above).

2.1.2 Cube Root Extraction Algorithm from Levi ben Gershon's *Maaseh Hoshev*.

In the same section, Levi also discusses a method for extracting cube roots. He does this with a similar algorithm centering around the identity $(a + b)^3 = a^3 + 3ab(a+b) + b^3$, (Theorem 62 from Part 1 of *Maaseh Hoshev*), in place of the identity $(a+b)^2 = a^2 + b^2 + 2ab$, (Theorem 6 from Part 1 of *Maaseh Hoshev*).

Given N , Levi constructs the cube root as he did for the square root, digit by digit, in order from the most significant to the least significant. Let a be the cube root constructed so far. Levi says to let the next digit be $\lfloor (N - a^3)/(3a(a+1)) \rfloor$. This approximation is based on the fact that $N \approx a^3 + 3ab(a+b)$, when $b \ll a$. However, it sometimes makes b too large. Levi warns to be careful that:

1. There is enough leftover to account for the b^3 , and
2. The ratio of $N - a^3 - 3ab(a+1) - b^3$ to $N - a^3$ is bigger than the ratio of $b - 1$ to $a + b$.

If the conditions are not met, he instructs us to subtract one from b , and try again to meet the conditions.

The first condition is similar to his warning in the square root algorithm, and ensures that the current cube root is not too large due to the extra b^3 . The strange second condition is similar. It is needed to account for the fact that $3a(a+1)$ is smaller than $3a(a+b)$. After b is added to the current cube root, Levi wants $N > (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3 = a^3 + 3ab(a+b) + b^3$. Levi gives the identity

$$3ab(a+b) = 3ab(b-1) + 3ab(a+1),$$

and so equivalently he wants $N - a^3 - 3ab(a+1) - b^3 > 3ab(b-1)$. This last condition is the same as his required second condition when $N - a^3 \approx 3ab(a+b)$.

To prime the pump, Levi gets the first approximation of the cube root of N by examining a left prefix of N of at most three digits. (Note that Levi writes his numbers in left to right order from most significant to least significant digit, as we do, despite the fact that the flow of the Hebrew around it is going right to left. A note of interest to Hebrew readers is that Levi uses the first nine Hebrew letters to stand for the digits 1-9, and does not use the standard Hebrew numbering scheme unless the numbers are smaller than 1000.)

To decide whether the prefix is one, two or three digits, he numbers the digits of N from right to left, in increasing order starting with 1. The prefix desired is the shortest prefix that ends in a digit numbered $3x+1$. He instructs us to use find the largest cube smaller than this prefix, and place its cube root in column $x+1$, as the first digit.

Example of Levi's Cube Root Extraction Algorithm

Levi considers the example 654321.

He considers the prefix 654, and therefore places 8 as the first digit of the cube root in column 2. Hence the first approximation of the cube root is $a = 80$. He then subtracts 512000 from 654321 to get 142321. He approximates the next digit by dividing 142321 by $(3)(80)(81)$, and ignoring the remainder to get 7. Then he checks his two conditions. For the first condition he calculates $142321 - 7(3)(80)(81) - 7^3 = 5894$, and notes that it is greater than zero. Then he notes however, that $5894/142321$ is not greater than $6/87$. Hence he instructs us to try 6 instead of 7. Repeating the checks of the two conditions, he gets that $142321 - 6(3)(80)(81) - 6^3 = 25465$, and that $25465/142321 > 5/86$. Then he goes ahead and subtracts $6(3)(80)(86) + 6^3$ from 142321 to get 18265. As an aside he notes, that due to the identity $3ab(a+b) = 3ab(b-1) + 3ab(a+1)$, an easier way to calculate 18265, is to simply subtract $3(80)(5)(6)$ from 25465. At this point Levi notes that it is impossible to get any closer to the cube root with integers because adding one to the original cube root of 80, adds at least $3(80)(81) = 19440$, which is already more than what we have currently leftover, that is 18265.

86
654321
512

142321
124056

18265

An Easier Way

Levi notes that it may be cumbersome to do the calculations required to make sure that his two conditions hold, and thereby ensure that the next digit is not too high. To this end, he gives an alternate *easier* method, guaranteed never to generate digits that are too high. That is, he guarantees that at any stage of the algorithm, $(a+b)^3 < N$, where a is the current cube root, b is the new amount added, and N is the original number. Note that this new method, in contrast to the previous one, does not guarantee that the digit will be the largest possible. The previous method guarantees a lower bound on the new digit, and this method ensures an upper bound.

Let the last digit of a be in the i th column from the right. He instructs us to let the next digit equal the current leftover divided by $1/10$ of $((a+10^{i-1})^3 - a^3)$. In the previous example the current leftover is 142321, $i = 2$, and $1/10$ of $(90^3 - 80^3) = 21700$, therefore the next digit equals $142321/21700 \approx 6$. In general, this shortcut sets the next digit equal to the integer part of:

$$(N-a^3)/(3a^2 10^{i-2} + 3a 10^{2i-3} + 10^{3i-4}).$$

Since $b = 10^{i-2}$ times this digit, this implies that $N = a^3 + 3ba^2 + 3ba 10^{i-1} + b 10^{2i-2}$. Note that $(a+b)^3 < N$ when $b < 10^{i-1}$. But b is always smaller than 10^{i-1} , since it begins with the $(i-2)$ th digit, hence the next digit calculated with this method is never too high.

Note Levi does not claim that this new *easier* method always gives the largest possible digit. In fact, it does not. For example, the cube root of 6859, is calculated with 10 as the starting approximation. This leaves over 5859, and according to this method, $(20^3 - 10^3)/10 = 700$, and so the second digit equals $5859/700 \approx 8$, rather than the correct value of 9. Hence Levi simply provides two methods, one that gives an upper bound on the next digit, and one that gives a lower bound, letting the reader choose.

Levi's algebraic fluency is indicated by his manipulation of the various polynomial identities which form the basis of this cube root algorithm. It is distinct from the geometric approach of his predecessors. We see this same algebraic focus in the next two sections.

2.1.3 An Example from the Problem Section of *Maaseh Hoshev*

A critical edition of all 21 problems in this section, including details on the two different editions of *Maaseh Hoshev* can be found in [Si00a]. Here we present a problem on certain simultaneous linear equations that shows Levi's algebraic sophistication. Given A , B and C , find X , Y and Z , such that $X+(Y+Z)/A = Y + (X+Z)/B = Z + (X+Y)/C$.

He gives the solution as: $X = C + (B-A) + (A-2)BC$, $Y = X + 2(B-A)(C-1)$ and $Z = Y + 2(A-1)(C-B)$. Note that there are in fact an infinite number of solutions to this problem. Levi understands this and explains that these three values are the “basic” solution. He continues to explain that the presence of an additional constraint, like the value of any of X , Y or Z , or the sum of any two, etc., determines a unique solution. This unique solution is determined by setting up the appropriate proportions with the basic solution. His proof is long and tedious, and is based on theorems 44-52 from Part 1 of *Maaseh Hoshev*.

44. $ab+a = (b+1)a$, and $ab+b = (a+1)b$.
45. Given $a < b < c$, then $c(b-a) + a(c-b) = b(c-a)$.
46. Given $2 < a < b$, then $2(a-2)(b-1) + b + (a-2) + (b-a) = 2(a-1)(b-1)$.
47. Given $a < b$, then $ba + (b-a) = (a-1)(b-1) + b + (b-1)$.
48. Given $2 < a < b$, then $2(b-1)(a-2) + b + (a-2) + (b-a) = (a-1)b + (a-2)(b-1) + (b-a)$.
49. Given $a < b < c$, and $d = a-2$, then $2(c-1)(b-a) + cd + (c-1)d + c + (b-a) + (c-b) = 2(b-1)(c-1)$.
50. Given $a < b < c$, and $d = a-2$, then $2(c-1)(b-a) + (cb)d + (c-b)(a-1) + c + (b-a) = (b-1)(c-1)a$.
51. Given $a < b < c$, then $(c-1)(b-a) + c + (b-a) = c(b-a+1)$.
52. Given $a < b < c$, then $(c-1)(b-a) + (a-1)(c-b) + c + (b-a) = b(c-a+1)$.

2.1.4 Levi’s Algebraic and Combinatorial Identities

In the first part of *Maaseh Hoshev*, Levi prepares the reader for the algorithms in Part 2, with a collection of theorems. Many of these show Levi’s algebraic bag of tricks. Some of these theorems are shown in the last section, and are used in the solution to Problem 21. Other theorems include formulae for sums of consecutive integers, squares and cubes. In the last case, he makes elegant use of mathematical induction [Ra70]. He also has a complete proof of some fundamental combinatorial identities. A list of these theorems with brief commentary in parentheses, appears below. Of course, Levi has no notation for these theorems and so they are all heavily worded. For example, Theorem 27 looks like this:

When you add consecutive numbers starting with one, and the number of numbers is odd, the result is equal to the product of the middle number among them, times the last number.

And Theorem 65 looks like this:

When there is a given number of different elements and the number of permutations of a second given number of these elements selected from the first given number, and smaller than it, changing in the order and in their elements, is the third given number, then the number of permutations of the successor to the second number from these elements is the product of the third given number by the excess of the first number over the second number.

Algebraic Identities

26. $1+2+\dots+n$, where n is even, is equal to $(n/2)(n+1)$.

(Literally, half the number of terms times the number of terms plus 1. Proof works from outside in, in pairs showing that each pair sums to $n+1$, and there are $n/2$ pairs).

27. $1+2+\dots+n$, where n is odd, is equal to $((n+1)/2)n$.
(Literally the middle term times the number of terms. Proof works from inside out, in pairs showing that each pair sums to twice the middle term.)
28. $1+2+\dots+n$, where n is odd, is equal to $(n/2)(n+1)$.
(Literally, half the last term times the number after the last term.
Proof uses proportions, algebra like idea and 21).
29. $1+3+5+\dots+(2n-1) = n^2$.
(Literally, the square of the middle term. Proves it first for an odd number of terms, then an even number).
30. $(1+2+\dots+n)+(1+2+\dots+n+(n+1)) = (n+1)^2$.
31. $2(1+2+\dots+n) = n^2+n$.
(Proof uses 30).
(Corollary: The sum $1+2+\dots+n = n^2/2 + n/2$.)
32. $1 + (1+2) + (1+2+3) + \dots + (1+2+\dots+n) = 2^2+4^2+6^2+\dots+n^2$, n even; and $1^2+3^2+5^2+\dots+n^2$, n odd.
(Proof uses 30).
33. $(1+2+3+\dots+n)+(2+3+4+\dots+n)+ \dots + n = 1^2+2^2+3^2+\dots+n^2$
(Proof uses a counting argument).
34. $(1+2+3+\dots+n)+(2+3+4+\dots+n)+ \dots + n+1+(1+2)+(1+2+3) + \dots + (1+2+\dots+(n-1)) = n(1+2+3+\dots+n)$
(Proof uses a counting argument).
35. $(n+1)^2+n^2 - (n+1+n) = 2n^2$
36. $(1+2+3+\dots+n)+(2+3+4+\dots+n)+ \dots + n - (1+2+3+\dots+n) = 2(2^2+4^2+6^2+\dots+(n-1)^2)$, $n-1$ even; and $2(1^2+3^2+5^2+\dots+(n-1)^2)$, $n-1$ odd.
(Proof uses 33 and 35).
37. $n(1+2+3+\dots+(n+1)) = 3(1^2+3^2+5^2+\dots+n^2)$, n odd; and $3(2^2+4^2+6^2+\dots+n^2)$, n even.
(Proof uses 32, 34 and 36).
38. $(n-(1/3)(n-1))(1+2+3+\dots+n) = 1^2+2^2+3^2+\dots+n^2$
(Proof uses 32, 33, 34 and 37).
39. $(n^2-n)/2 = (1+2+\dots+(n-1))$
(Proof uses 30).
40. $(n^2-n)/2 + n = (1+2+\dots+n)$
(Proof uses 30).
41. $(1+2+3+\dots+n)^2 = n^3 + (1+2+3+\dots+(n-1))^2$
(Proof uses 30 and 6).
42. $(1+2+3+\dots+n)^2 = 1^3+2^3+3^3+\dots+n^3$
(Proof by induction using 41).
43. Let $m = 1+2+3+\dots+n$, then $1^3+2^3+3^3+\dots+n^3 = 1+3+5+\dots+(2m-1)$.

Combinatorial Identities

63. $P_{n+1} = (n+1)P_n$, where P_n is the number of different ways to order n elements.
(Corollary: $P_n = n!$).
64. $P_{n,2} = n(n-1)$, where $P_{n,m}$ is the number of ways to order m elements out of n .
65. $P_{n,m+1} = P_{n,m} (n-m)$.
(Corollary: $P_{n,m} = n!/(n-m)! = n(n-1)(n-2)\dots(n-(m+1))$).
66. $P_{n,m} = C_{n,m}P_m$, where $C_{n,m}$ is the number of ways to choose m elements out of n without regard to order.
67. $C_{n,m} = P_{n,m}/P_m$
68. $C_{n,n-m} = C_{n,m}$.

2.2 Abraham ben Meir Ibn Ezra's *Sefer Hamispar*

Abraham ben Meir Ibn Ezra (1090-1167), mathematician, philosopher, grammarian and biblical commentator, lived in Spain two hundred years before Levi. Ibn Ezra does not prove his results in a rigorous Euclidean fashion like Levi, nor does he show any explicit use of mathematical induction, however he does present some of the same knowledge of algebraic identities. He was not as original a mathematical thinker as Levi, but more of a careful organizer who presented the known mathematics of his day. Although he was not the mathematician that Levi was, he too showed early signs of an algebraic tradition in the process of extrication from geometric notions. He exhibits certain combinatorial theorems in his works on astrology while discussing the conjunctions of planets [Ka93]. However, his only purely mathematical work is *Sefer Hamispar*, in many ways a primitive version of Part 2 of *Maaseh Hoshev*.

Sefer Hamispar, 1146, is a book on arithmetic and calculation in seven sections with a short introduction explaining the decimal system and the use of 0 as a place keeper. Note that both Ibn Ezra and Levi use the decimal system for integers but base 60 for fractions, as was common in that day.

- Section 1 - Multiplication.
- Section 2 - Division.
- Section 3 - Sums.
- Section 4 - Differences.
- Section 5 - Fractions.
- Section 6 - Ratios.
- Section 7 - Square roots.

2.2.1 Square Root Extraction Algorithm of Ibn Ezra from *Sefer Hamispar*

Ibn Ezra has no algorithm for cube root extraction, but does discuss a method for square roots based implicitly on the identity $(a+b)^2 = a^2 + b^2 + 2ab$, and the related approximation $\sqrt{x^2 \pm A} = x \pm A/2x$. This identity and approximation were known to the Babylonians and certainly the Greeks, and have very simple geometric interpretations, similar to those discussed in 2a. Ibn Ezra, however, makes no reference to the geometric viewpoint.

Unlike Levi who provided careful proofs, general examples and special cases, Ibn Ezra's style is to explain his ideas through numerous examples, relying on quantity to substitute for rigor. Here are two of his examples for calculating the closest integer square root of a given integer.

$$\sqrt{7500} = \sqrt{(8100 - 600)} \approx 90 - 600/180 \approx 86.$$

$$\sqrt{600000} = \sqrt{(640000 - 40000)} \approx 800 - 40000/1600 \approx 774.$$

He checks his calculations by squaring the answers implicitly using the identity $(a-b)^2 = a^2 + b^2 - 2ab$. For example, $(800-26)^2$ gives $640000 - 51600 + 26^2 = 587724$.

2.2.2 Identities

Through many examples he shows how to use the identities $ab = \sqrt{a^2b^2}$, $a/b = \sqrt{a^2/b^2}$, $(a/b)^2b^2 = a^2$, and $(a \pm b)^2 = a^2 + b^2 \pm 2ab$ in order to break larger problems up into smaller ones. For example, he shows how to calculate 19^2 if you know 7^2 , by calculating $(19/7)^2 = (2+5/7)^2 = 2^2 + 2(2)(5/7) + (5/7)^2 = 4 + 20/7 + 25/49 = 7 + 2/7 + 4/49$, and then multiplying by 7^2 to get $343 + 14 + 4 = 361$.

In Section 3 on sums, he explicitly states the formula for the sum of the first n integers in two different ways similar to Levi's Theorems 26 - 28. He adds what he claims is his own formula, which is the same as Levi's Theorem 31. He also states the formula for the sum of the first n squares, Levi's Theorem 38, and he adds the well known formula for summing geometric series.

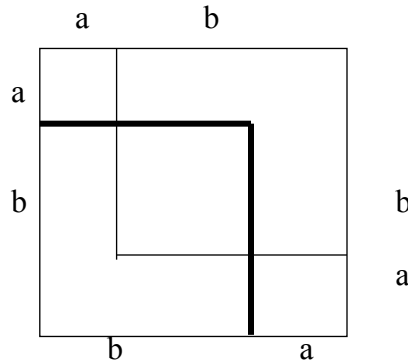
A Nontrivial Identity

He concludes the section on square roots with a nontrivial identity, showing that indeed he had a sophisticated sense of algebra. This identity is shown by examples of special cases followed by the general case, albeit no proof is given. Furthermore, this identity is not to be found in Levi's work.

He starts with the identity $2(a^2+b^2) - (a-b)^2 = (a+b)^2$. Then he generalizes this to $3(a^2+b^2+c^2) - (a-b)^2 - (b-c)^2 - (a-c)^2 = (a+b+c)^2$. He then states the case for four and finally states the general theorem:

The square of the sum of any number of integers equals the sum of the squares, times the number of integers in your sum, minus the squares of the differences between all pairs of integers in your sum.

The first special case, $2(a^2+b^2) - (a-b)^2 = (a+b)^2$, is easily thought of geometrically. One way is the figure below:



It is not as easy to imagine how this could be completely generalized geometrically. However it is easy to prove this theorem by induction, using just the simple identities available to Ibn Ezra. In the equations that follow, we assume that the sums over all i and j are from 1 to n . We wish to prove that:

$$(*) \quad n\sum a_i^2 = (\sum a_i)^2 + \sum (a_i - a_j)^2$$

Note that from $(a \pm b)^2 = a^2 + b^2 \pm 2ab$, one can derive:

$$(**) \quad (n+1)a_{n+1}^2 + \sum a_i^2 = \sum (a_{n+1} - a_i)^2 + a_{n+1}^2 + 2a_{n+1} \sum a_i$$

Adding both sides of (**) to both sides of (*), gives us the $(n+1)$ st case of the general theorem.

Ibn Ezra concludes by explaining that the number of these differences equals the sum of the consecutive numbers from 1 up to and including one less than number of integers in your sum. That is, he is well aware of the identity:

$$C(n,2) = 1+2+3+\dots+(n-1).$$

2.3 Conclusion

The work of Medieval Hebrew mathematicians shows the progression of algebra from a geometric subject to a symbolic one. Through algorithms, creative puzzles, inductive arguments and algebraic identities, their work shows an emphasis on the combinatorial aspects of algebra.

Bibliography

- [CP92] Chemla, Karine and Pahaut, Serge, *Remarques sur les Ouvrages Mathematiques de Gersonide*, in Gad Freudenthal (ed.), **Studies on Gersonides-A 14th Century Jewish Philosopher-Scientist**, Leiden: Brill, 149-191 (1992).
- [EJ71] Encyclopedia Judaica, Volume 11, Macmillan and Company. 91-94 (1971).
- [Fr88] Freudenthal, Gad, "Maimonides' 'Guide of the Perplexed' and the Transmission of the Mathematical Tract 'On Two Asymptotic Lines' in the Arabic, Latin and Hebrew Medieval Traditions." *Vivarium* 26: 113-140, 1988.
- [Go85] Goldstein, B. R., "Scientific Traditions in Late Medieval Jewish Communities." In *Les Juifs au regard de l'histoire. Mélanges en l'honneur de Bernhard Blumenkranz*, G. Dahan, ed. Paris, 1985.
- [Ka93] Katz, Victor J., **A History of Mathematics: An Introduction**, Harper Collins, (1993).
- [Ka96] Katz, Victor J., *Combinatorics and Induction in Medieval Hebrew and Islamic Mathematics*, in Ron Calinger (editor), **Vita Mathematica: Historical Research and Integration with Teaching**, Mathematical Association of America, 99-107 (1996).
- [Ke92] Kellner, Menachem, *An Annotated List of Writings by and about R. Levi ben Gershom*, in Gad Freudenthal (ed.), **Studies on Gersonides-A 14th Century Jewish Philosopher-Scientist**, Leiden: Brill, 367-414 (1992).
- [La09] Lange, Gerson, *Sefer Maassei Chosheb-Die Praxis des Rechners, Ein hebraeisch arithmetisches Werk des Levi ben Gerschom aus dem Jahre 1321*, Frankfurt am Main: Louis Golde, (1909).
- [LH84] Langermann, Y. T., and Hogendijk, J. P., "A Hitherto Unknown Hellenistic Treatise on the Regular Polyhedra," *Historia Mathematica* 11:325-326, 1984.
- [La84a] Langermann, Y. T., "The Mathematical Writings of Maimonides." *Jewish Quarterly Review* 75: 57-65, 1984.
- [La84b]--, "Sefer Uqlidis' by an Anonymous Author" [Hebrew], *Kiryat Sefer* 54:635, 1984.
- [La88] --"The Scientific Writings of Mordekhai Finzi", *Italia: Studie ricerche sulla storia, la cultura e la letteratura degli ebrei d'Italia*, 7: 7-44,1988.
- [La96] --"Medieval Hebrew Texts on the Quadrature of the Lune," *Historia Mathematica* 23:31-53, 1996.

- [Le52] Levey, Martin. "The Encyclopaedia of Abraham Savasorda: A Departure in Mathematical Methodology." *Isis* 43: 257-264, 1952.
- [Le54] --"Abraham Savasorda and His Algorism: A Study in Early European Logistic." *Osiris* 11: 50-64, 1954.
- [Lé89a]Lévy, Tony, "L'étude des sections coniques dans la tradition médiéval hébraïque: ses relations avec les traditions arabe et latine." *Revue de l'Histoire des Sciences* 42: 193-239, 1989.
- [Lé89a]-- "Le Chapitre I, 73 du *Guide des Égarés* et la tradition mathématique hébraïque au moyen âge: un commentaire in''dit de Salomon b. Isaac." *Revue des Études juives* 147: 307-336, 1989.
- [Lé92]--"Gersonide, commentateur d'Euclide: Traduction annotée de ses gloses sur les *Éléments*." In *Studies on Gersonides, a Fourteenth Century Jewish Philosopher-Scientist*, Gad Freudenthal, ed. Leiden: Brill, 1992.
- [Lé96]--"Hebrew Mathematics in the Middle Ages: An Assessment." In *Tradition, Transmission, Transformation*, F. J. Ragep and S. Ragep, eds. Leiden: Brill, 1996.
- [Lé97] --"The Establishment of the Mathematical Bookshelf of the Medieval Hebrew Scholar: Translations and Translators," *Science in Context* 10 (3): 431-451, 1997.
- [Mu20] Müller, Ernst. *Abraham Ibn Esra. Buch der Einheit*. Berlin: Welt-Verlag, 1920.
- [Ra70] Rabinovitch, Nahum L., *Rabbi Levi ben Gershon and the Origins of Mathematical Induction*, **Archive for History of Exact Sciences**, 6, 237-248 (1970).
- [Si1895] Silverberg, Moshe, *Sefer Hamispar of Rabbi Abraham ibn Ezra*, Frankfurt am Main: Y. Kaufmann (1895).
- [Si00a] Simonson, Shai, *The Missing Problems of Gersonides, A Critical Edition Parts I and II*, *Historia Mathematica*, to appear.
- [Si00b] Simonson, Shai, *Mathematical Gems of Levi ben Gershon*, *Mathematics Teacher*, to appear.
- [Si00c] Simonson, Shai, *The Mathematics of Levi ben Gershon, the Ralbag*, **BDD 9**, Bar-Ilan University Press, 2000.
- [St1893] Steinschneider, M. *Die hebraeische Uebersetzungen des Mittelalters und die Juden als Dometscher*. Berlin, 1893.
- [St64] --, 1893-1901, *Die Mathematik bei den Juden* (reprint, Hildesheim, 1964). Sternberg, Shlomo, ed. *Studies in Hebrew Astronomy and Mathematics by Solomon Gandz*. New York: Ktav, 1970.

