The Fundamental Group of the Circle

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May 1, 2007

Abstract

In this paper, we use covering spaces to prove that the fundamental group of the circle is the set of integers.

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1 Introduction

In this paper, we use covering spaces to prove that the fundamental group of the circle is the set of integers.

2 Preliminary

Definition 2.1 The circle is the subset of $\mathbb{R}^2$ which consists the pairs $(x, y)$ such that $x^2 + y^2 = 1$. It is denoted by $S^1$.

Definition 2.2 Given points $x$ and $y$ of the space $X$, a path in $X$ from $x$ to $y$ is a continuous map $f : [0, 1] \rightarrow X$ such that $f(0) = x$ and $f(1) = y$. 
Definition 2.3 A space $X$ is said to be **path-connected** if every pair of points of $X$ can be joined by a path in $X$.

Definition 2.4 If $f$ and $f'$ are continuous maps of the space $X$ into the space $Y$, we say that $f$ is **homotopic** to $f'$ is there is a continuous map $F : X \times I \to Y$ such that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = f'(x)$$

for each $x$. (Here $I = [0,1]$.) The map $F$ is called a **homotopy** between $f$ and $f'$.

Notation 2.5 If $f$ is homotopic to $f'$ we write $f \simeq f'$.

Definition 2.6 If $f \simeq f'$ and $f'$ is a constant map, we say that $f$ is **nullhomotopic**.

Definition 2.7 Two paths $f$ and $f'$, mapping the interval $I = [0,1]$ into $X$, are said to be path homotopic if they have the same initial point $x_0$ and the same final point $x_1$ and if there is a continuous map $F : I \times I \to X$ such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = f'(x),$$
$$F(0, t) = x_0 \quad \text{and} \quad F(1, t) = x_1,$$

for each $s, t \in I$. We call $F$ a **path homotopy** between $f$ and $f'$.

Notation 2.8 If $f$ is path homotopic to $f'$ we write $f \simeq_p f'$.

Lemma 2.9 ([Munkres] Lemma 51.1) The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Definition 2.10 If $f$ is a path in $X$ from $x_0$ to $x_1$, and if $g$ is a path in $X$ from $x_1$ to $x_2$, we define the product $f \ast g$ of $f$ and $g$ to be the path $h$ given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0, \frac{1}{2}] \\ g(2s - 1) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}.$$
2.1 Fundamental Group

Definition 2.11 Let \( X \) be a space and \( x_0 \) be a point of \( X \). A path in \( X \) that begins and ends at \( x_0 \) is called a loop based at \( x_0 \).

Proposition 2.12 Let \( X \) be a space and \( x_0 \) be a point of \( X \). The set of path homotopy classes of loops bases at \( x_0 \), with the operation \(*\), is a group, called the fundamental group of \( X \) relative to the base point \( x_0 \) and denoted by \( \pi_1 (X, x_0) \).

Definition 2.13 A space \( X \) is said to be simply connected if it is a path-connected space and if \( \pi_1 (X, x_0) \) is the trivial (one-element) group for some \( x_0 \in X \).

3 The Main Result

3.1 Covering Spaces

Definition 3.1 Let \( p : E \to B \) be a continuous surjective map. The open set \( U \) of \( B \) is said to be evenly covered by \( p \) is the interse image \( p^{-1} (U) \) can be written as the union of disjoint open sets \( V_\alpha \) in \( E \) such that for each \( \alpha \), the restriction of \( p \) to \( V_\alpha \) is a homeomorphism of \( V_\alpha \) onto \( U \).

Definition 3.2 Let \( p : E \to B \) be a continuous surjective map. If every point \( b \) of \( B \) has a neighborhood \( U \) that is evenly covered by \( p \), then \( p \) is called a covering map, and \( E \) is said to be a covering space of \( B \).

Theorem 3.3 ([Munkres] Theorem 53.1) The map given by the equation

\[
p (x) = (\cos 2\pi x, \sin 2\pi x)
\]

is a covering map.

3.2 The Fundamental Group of the Circle

Definition 3.4 Let \( p : E \to B \) be a map. If \( f \) is a continuous mapping of some space \( X \) into \( B \), a lifting of \( f \) is a map \( \tilde{f} : X \to E \) such that \( p \circ \tilde{f} = f \).
Remark 3.5 We use the following diagram to represent the concept of lifting.

\[ \begin{array}{ccc}
E & \xrightarrow{\tilde{f}} & B \\
\downarrow p & & \downarrow p \\
X & \xrightarrow{f} & B
\end{array} \]

Definition 3.6 Let \( p : E \to B \) be a covering map and \( b_0 \in B \). Choose \( e_0 \) so that \( p(e_0) = b_0 \).

Given an element \([f]\) of \( \pi_1(B, b_0) \). Let \( \tilde{f} \) be the lifting of \( f \) to a path in \( E \) that begins at \( e_0 \). Let \( \phi([f]) \) denote the end point \( \tilde{f}(1) \) of \( \tilde{f} \). Then \( \phi \) is well-defined set map

\[ \phi : \pi_1(B, b_0) \to p^{-1}(b_0). \]

We call \( \phi \) the lifting correspondence derived from the covering map \( p \).

Theorem 3.7 ([Munkres] Theorem 54.4) Let \( p : E \to B \) be a covering map and \( p(e_0) = b_0 \), where \( e_0 \in E \) and \( b_0 \in B \) are the base points. If \( E \) is path connected, then the lifting correspondence

\[ \phi : \pi_1(B, b_0) \to p^{-1}(b_0) \]

is surjective. If \( E \) is simply connected, it is bijective.

Proof. If \( E \) is path connected, then, given \( e_1 \in p^{-1}(b_0) \), there is a path \( \tilde{f} \) in \( E \) from \( e_0 \) to \( e_1 \). Then \( f = p \circ \tilde{f} \) is a loop in \( B \) at \( b_0 \), and \( \phi([f]) = e_1 \) by definition. Thus, \( \phi \) is surjective.

Suppose that \( E \) is simply connected. Let \([f]\) and \([g]\) be two elements of \( \pi_1(B, b_0) \) such that \( \phi([f]) = \phi([g]) \). Let \( \tilde{f} \) and \( \tilde{g} \) be the liftings of \( f \) and \( g \), respectively, to paths in \( E \) that begin at \( e_0 \). Then \( \tilde{f}(1) = \tilde{g}(1) \). Since \( E \) is simply connected, there is a path homotopy \( \tilde{F} \) in \( E \) between \( \tilde{f} \) and \( \tilde{g} \). Then \( p \circ \tilde{F} \) is a path homotopy in \( B \) between \( f \) and \( g \). This means that \( f \simeq g \), or \([f] = [g]\). Thus, \( \phi \) is injective. Therefore, \( \phi \) is bijective.

Theorem 3.8 The fundamental group of \( S^1 \) is isomorphic to the additive group of integers.

Proof. Let \( p : \mathbb{R} \to S^1 \) be the covering map of Theorem 3.3. Let \( e_0 = 0 \), and let \( b_0 = p(e_0) \).

Then \( p^{-1}(b_0) \) is the set \( \mathbb{Z} \) of integers. Since \( \mathbb{R} \) is simply connected, the lifting correspondence

\[ \phi : \pi_1(S^1, b_0) \to \mathbb{Z} \]
is bijective by the theorem 3.7. So, if we can show that $\phi$ is a group homomorphism, then the theorem is proved.

Given $[f]$ and $[g]$ in $\pi_1(S^1, b_0)$. Let $\tilde{f}$ and $\tilde{g}$ be their respective lifting to paths on $\mathbb{R}$ beginning at 0. Let $n = \tilde{f}(1)$ and $m = \tilde{g}(1)$. Then, $\phi([f]) = n$ and $\phi([g]) = m$ by the definition of $\phi$. Let $h$ be the path

$$h(s) = n + \tilde{g}(s)$$

on $\mathbb{R}$. Because $p(n + x) = p(x)$ for all $x \in \mathbb{R}$, the path $h$ is a lifting of $g$ which begins at $n$. Thus, the product $\tilde{f} \ast h$ is defined, and it is the lifting of $f \ast g$ that begins at 0. The end point of this path is $h(1) = n + m$. Then, by definition,

$$\phi([f] \ast [g]) = n + m = \phi([f]) + \phi([g]).$$

Therefore, $\phi$ is a group homomorphism. ■

References
