A Survey of the BP Theory

Hsin-hao Su

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In this article, $\longrightarrow\longrightarrow\longrightarrow$, $\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow$ and $\begin{array}{c}\downarrow\\ \downarrow\\ \downarrow\end{array}$ all mean a long arrow.

1 Preliminary

In this section, we state some useful definitions, terminologies, and propositions. Let A_* be the dual Steenrod algebra and $A_* = \mathbb{Z}/2\left[\xi_1, \xi_2, \cdots\right]$, where $\deg \xi_i = 2^i - 1$.

Let $m_A : A_* \otimes A_* \to A_*$ be the multiplication of A_*.

Proposition 1.1 *Let Δ be the coproduct of A_*, i.e., $\Delta : A_* \to A_* \otimes A_*$. Then $\Delta\left(\xi_n\right) = \sum\limits_{0 \le i \le n} \xi_{n-i}^{2^i} \otimes \xi_i$.*

Proof. See [Milnor1958]. ∎

Let E be the exterior algebra of A_*, i.e., $E = \mathbb{Z}/2\left[\xi_1, \xi_2, \cdots\right] \big/ \left(\xi_i^2\right)$. We have a natural projection $p_E : A_* \longrightarrow E$. By combining p_E and all operations of A_*, we can admit that E is a Hopf algebra.

2 The Thom Spectrum MU

Let MU be the Thom spectrum.

Since MU is a ring spectrum, we have a multiplication, $m_{MU} : H_*\left(MU; \mathbb{Z}/2\right) \otimes H_*\left(MU; \mathbb{Z}/2\right) \to H_*\left(MU; \mathbb{Z}/2\right)$.

We know that $H_*\left(\mathbb{C}P^\infty; \mathbb{Z}/2\right) \cong \mathbb{Z}/2\left[y_1, y_2, \cdots\right]$, where $\deg y_i = 2i$. And, there is a map $C : \sum^{-2} \mathbb{C}P^\infty \longrightarrow MU$.

Proposition 2.1 *$H_*(MU;\mathbb{Z}/2) \cong \mathbb{Z}/2[b_1, b_2, \cdots]$, where $\deg b_i = 2i$.*

According to Switzer's book[SwitzerBook1], we have the following Switzer formula.

Proposition 2.2 *Let $\psi_{\mathbb{C}P^\infty}$ be the left A_*-coaction of $H_*(\mathbb{C}P^\infty;\mathbb{Z}/2)$. Then we have $\psi_{\mathbb{C}P^\infty}(y_n) = \sum_{i=0}^{n}\left[(\xi)^i_{n-i}\right]^2 \otimes y_i$, where $\xi = 1+\xi_1+\xi_2+\cdots$.*

Proposition 2.3 *Let ψ_{MU} be the left A_*-coaction of $H_*(MU;\mathbb{Z}/2)$. Then we have $\psi_{MU}(b_n) = \sum_{i=1}^{n+1}\left[(\xi)^i_{n+1-i}\right]^2 \otimes b_{i-1}$, where $\xi = 1+\xi_1+\xi_2+\cdots$.*

Proof. By the computation of $H_*(MU;\mathbb{Z}/2)$, we get $C_*(y_{n+1}) = b_n$ for all n. We have the following commutative diagram

$$\begin{array}{ccc} H_*(\mathbb{C}P^\infty;\mathbb{Z}/2) & \xrightarrow{\psi_{\mathbb{C}P^\infty}} & A_*\otimes H_*(\mathbb{C}P^\infty;\mathbb{Z}/2) \\ C_* \downarrow & & \downarrow 1\otimes C_* \\ H_*(MU;\mathbb{Z}/2) & \xrightarrow{\psi_{MU}} & A_*\otimes H_*(MU;\mathbb{Z}/2) \end{array}.$$

Therefore,

$$\begin{aligned} \psi_{MU}(b_n) &= \psi_{MU}(C_*(y_{n+1})) \\ &= (1\otimes C_*)\circ\psi_{\mathbb{C}P^\infty}(y_{n+1}) \\ &= (1\otimes C_*)\left(\sum_{i=0}^{n+1}\left[(\xi)^i_{n+1-i}\right]^2\otimes y_i\right) \\ &= \sum_{i=1}^{n+1}\left[(\xi)^i_{n+1-i}\right]^2\otimes b_{i-1}. \end{aligned}$$

■

3 Brown-Peterson Algebraic Splitting

Let $P = \mathbb{Z}/2\left[\bar{b}_i | i \neq 2^l - 1\right]$. We define $f : H_*(MU;\mathbb{Z}/2) \longrightarrow P$ by

$$f(b_n) = \begin{cases} \bar{b}_n & \text{, if } n \neq 2^l-1 \text{ for all } l \\ 0 & \text{, if } n = 2^l-1 \text{ for some } l \end{cases}$$

and $\bar{f}$ is defined as the following composite map

$$H_*(MU;\mathbb{Z}/2) \xrightarrow{\psi_{MU}} A_* \otimes H_*(MU;\mathbb{Z}/2) \xrightarrow{1\otimes f} A_* \otimes P,$$

i.e., $\bar{f} = (1\otimes f)\circ\psi_{MU}$. By the multiplication of $H_*(MU;\mathbb{Z}/2)$, we can define the multiplication of P, denoted by m_P, as the following diagram

$$\begin{array}{ccc}
H_*(MU)\otimes H_*(MU) & \xrightarrow{m_{MU}} & H_*(MU) \\
f\otimes f \downarrow & & \downarrow f \\
P\otimes P & \xrightarrow{m_P} & P
\end{array}$$

We know that $\bar{f}$ is an algebra map by checking the commutativity of the following diagram

$$\begin{array}{ccccc}
H\otimes H & & \xrightarrow{m_{MU}} & & H \\
\psi_{MU}\otimes\psi_{MU} \downarrow & & & & \downarrow \psi_{MU} \\
A_*\otimes H\otimes A_*\otimes H & \xrightarrow{1\otimes T\otimes 1} & A_*\otimes A_*\otimes H\otimes H & \xrightarrow{m_A\otimes m_{MU}} & A_*\otimes H \\
1\otimes f\otimes 1\otimes f \downarrow & & 1\otimes 1\otimes f\otimes f \downarrow & & \downarrow 1\otimes f \\
A_*\otimes P\otimes A_*\otimes P & \xrightarrow{1\otimes T\otimes 1} & A_*\otimes A_*\otimes P\otimes P & \xrightarrow{m_A\otimes m_P} & A_*\otimes P
\end{array}$$

where H means $H_*(MU;\mathbb{Z}/2)$. And, in the following diagram

$$\begin{array}{ccc}
H_*(MU;\mathbb{Z}/2) & \xrightarrow{\psi_{MU}} & A_*\otimes H_*(MU;\mathbb{Z}/2) \\
\psi_{MU} \downarrow & \text{(A)} & \downarrow 1\otimes\psi_{MU} \\
A_*\otimes H_*(MU;\mathbb{Z}/2) & \xrightarrow{\Delta\otimes 1} & A_*\otimes A_*\otimes H_*(MU;\mathbb{Z}/2) \\
1\otimes f \downarrow & \text{(B)} & \downarrow 1\otimes 1\otimes f \\
A_*\otimes P & \xrightarrow{\Delta\otimes 1} & A_*\otimes A_*\otimes P
\end{array},$$

(A) commutes since $H_*(MU;\mathbb{Z}/2)$ is a A_*-comodule and (B) commutes clearly. So, $\bar{f}$ is a A_*-algebra map.

Lemma 3.1 *P is a A_*-algebra with a trivial coaction, that is, $\psi_P(b_n) = 1\otimes b_n$ for all n. In addition, P is an E-algebra and the E-coaction of P, named by ψ_P^E, is a trivial coaction.*

Proof. Consider P as a subalgebra of $A_* \otimes P$. By the above diagram, it is clear that P is a A_*-algebra. Since P has an extended A_*-comodule structure, it makes ψ_P a trivial coaction. Clearly, P is an E-algebra with trivial coaction. ■

Now, we are on the position to prove the Brown-Peterson algebraic splitting. Firstly, we prove a technical lemma.

Lemma 3.2 *Let M^k be the subalgebra of M generated by $1, \xi_1, \xi_2, \cdots, \xi_k$ and P^k be the subalgebra of P generated by $1, \bar{b}_1, \bar{b}_2, \cdots, \bar{b}_k$. Then we have*

1. *If $k = 2^l - 1$ for some l, then $\bar{f}(b_k) = \xi_l^2 \otimes 1 + X_1$, where $X_1 \in M^{k-1} \otimes P^{2^k-2}$.*
2. *If $2^{l-1} - 1 < k < 2^l - 1$ for some l, then $\bar{f}(b_k) = 1 \otimes \bar{b}_k + X_2$, where $X_2 \in M^{k-1} \otimes P^{k-1}$.*

Proof. It is true by expending Swizter formula. See [SwitzerBook1] lemma 20.6 in page 493. ■

Proposition 3.3 (Brown-Peterson) $H_*(MU; \mathbb{Z}/2) \cong M \otimes_{\mathbb{Z}/2} P$ *as A_*-algebra where $M = \mathbb{Z}/2\left[\xi_1^2, \xi_2^2, \cdots\right]$ is an A_*-subalgebra of A_* and $P = \mathbb{Z}/2\left[\bar{b}_i | i \neq 2^l - 1\right]$.*

Proof. Let $\bar{f}$ be defined as above. By the Switzer formula, we observe that $\operatorname{Im} \bar{f} \subseteq M \otimes P$. Therefore, $\bar{f} : H_*(MU; \mathbb{Z}/2) \longrightarrow M \otimes P$ is an A_*-algebra map.

As $\mathbb{Z}/2$-vector spaces, we have that $\dim H_*(MU; \mathbb{Z}/2) = \dim M \otimes P$, since both dimensions are finite and we have the following 1-1 correspondences

$$\begin{cases} b_n & \longleftrightarrow & 1 \otimes \bar{b}_n & \text{, for } n \neq 2^l - 1 \text{ for some } l \\ b_{2^l-1} & \longleftrightarrow & \xi_l^2 \otimes 1 & \end{cases},$$

in basis elements for counting dimensions. In proving $H_*(MU; \mathbb{Z}/2) \cong M \otimes_{\mathbb{Z}/2} P$ as $\mathbb{Z}/2$-vector spaces, it suffices to show that $\bar{f}$ is onto, i.e., $M \otimes P \subseteq \operatorname{Im} \bar{f}$. Of course, $M^0 \otimes P^0 \subseteq \operatorname{Im} \bar{f}$. For all t, s, we will prove $M^t \otimes P^s \subseteq \operatorname{Im} \bar{f}$ by induction on both indexes(See [SwitzerBook1] theorem 20.7 in page 493). Without loss of generality, we assume that $M^{i-1} \otimes P^{2^i-2} \subseteq \operatorname{Im} \bar{f}$ for some $i > 1$. We want to prove $M^i \otimes P^d \subseteq \operatorname{Im} \bar{f}$ for $2^i - 2 \leq d \leq 2^{i+1} - 2$ to complete our induction step. By lemma 3.2(1), we know that $M^i \otimes P^0 \subseteq \operatorname{Im} \bar{f}$. Assume that $M^i \otimes P^{j-1} \subseteq \operatorname{Im} \bar{f}$ for some $1 < j < 2^{i+1} - 1$. If $j = 2^d - 1$ for some $d > 1$ such that $1 \leq d \leq i$, then $P^j = P^{j-1}$ by its definition, that is, $M^i \otimes P^j \subseteq \operatorname{Im} \bar{f}$. Otherwise, $M^0 \otimes P^j \subseteq \operatorname{Im} \bar{f}$ by lemma 3.2(2). Since $\bar{f}$ is an A_*-algebra map, we conclude that $M^i \otimes P^j \subseteq \operatorname{Im} \bar{f}$ by using multiplication. This completes the induction step.

Combining two results in above, we conclude that $H_*(MU; \mathbb{Z}/2) \cong M \otimes_{\mathbb{Z}/2} P$ as A_*-algebra.

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4 Brown-Peterson Spectrum

Brown and Peterson first constructed a spectrum, BP, such that $H_*(BP) = \mathbb{Z}/2\left[\xi_1^2, \xi_2^2, \cdots\right]$. And Quillen used the multiplicative map and idempotent to construct a map g in the following

$$BP \longrightarrow MU_{(2)} \xrightarrow{g} MU_{(2)}.$$

5 To Compute the stable homptopy group of MU

We use the Adams spectral sequence to compute the $\pi_*(MU)$, the stable homotopy group of MU.

Proposition 5.1 *Let Q be a left A_*-comodule which is concentrated in even dimensions. Then Q is a comodule over M where $M = \mathbb{Z}/2\left[\xi_1^2, \xi_2^2, \cdots\right]$.*

Proof. Let ψ be the left coaction of Q. For all $q \in Q$, we assume $\psi(q) = \sum_k a_k \otimes q_k$, where $a_k \in A_*$ and $q_k \in Q$. Since $\deg q$ and $\deg q_k$ are all even, $\deg a_k$ must be even, i.e., a_k is represented by a multiplication of even number element in A_*. Assume that there exists an $a_i \in A_* \backslash M$, i.e., $a_i = a' \xi_i^k$, where a' does not consist by ξ_i and k is odd. Consider the coassociativity of ψ,

$$\begin{array}{ccc} P & \xrightarrow{\psi} & A_* \otimes P \\ \psi \downarrow & & \downarrow \Delta \otimes 1 \\ A_* \otimes P & \xrightarrow{1 \otimes \psi} & A_* \otimes A_* \otimes P \end{array}.$$

We have $(\Delta \otimes 1)(a_i \otimes q_i) = \Delta(a_i) \otimes q_i = \left(\sum_s m_s \otimes a_s\right) \otimes q_i$, where $m_s \in M$ and $a_s \in A_*$. By another way, we get $(1 \otimes \psi)(a_i \otimes q_i) = a_i \otimes \psi(q_i) = a_i \otimes \left(\sum_t a_t \otimes q_t\right)$, where $a_t \in A_*$ and $q_t \in Q$. But $a_i \notin M$, so we conclude that $(\Delta \otimes 1)(a_i \otimes q_i) \neq (1 \otimes \psi)(a_i \otimes q_i)$. Therefore, $a_i \in M$ for all i, i.e., $\psi(q) \in M \otimes Q$. Q is a comodule over M. ■

The E_2-term of the Adams spectral sequence to compute $\pi_*(MU)$ is

$$\mathrm{Ext}_{A_*}^{*,*}(\mathbb{Z}/2, H_*(MU; \mathbb{Z}/2)).$$

Before we determine it, we introduce a special case of the change-of-rings isomorphism theorem first. The cotensor product of A_* and P over E, denoted by $A_* \square_E P$, is the kernel of the following map

$$A_* \otimes_{\mathbb{Z}/2} P \xrightarrow{\Delta \otimes 1 - 1 \otimes \psi_P^E} A_* \otimes_{\mathbb{Z}/2} E \otimes_{\mathbb{Z}/2} P.$$

Proposition 5.2 *Let A_* be the dual Steenrod algebra and E be its exterior algebra. By the proposition proved in Section* **??**, *we know that $H_*(MU; \mathbb{Z}/2) \cong M \otimes_{\mathbb{Z}/2} P$ as A_*-algebra where M and P are defined as above. And we have*

$$\mathrm{Ext}_{A_*}^{*,*}(\mathbb{Z}/2, A_* \square_E P) \cong \mathrm{Ext}_{A_*}^{*,*}(\mathbb{Z}/2, P).$$

Proof. The proof of this theorem is just diagram chasing. See [SwitzerBook1] theorem 20.16 in page 498. ■

Corollary 5.3 *We have*

$$\mathrm{Ext}_{A_*}^{*,*}\left(\mathbb{Z}/2, A_* \otimes_{\mathbb{Z}/2} P\right) \cong \mathrm{Ext}_{E}^{*,*}(\mathbb{Z}/2, P) \cong \mathrm{Ext}_{E}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} P.$$

Proof. By the usual projection from A_* to E, we know that $\Delta(\xi_n) = \sum_{0 \leq i \leq n} \xi_{n-i}^{2^i} \otimes \xi_i$ for all n is the right E-comodule formula of the A_*. Obviously, we have $M \otimes_{\mathbb{Z}/2} P \subseteq \ker\left(\Delta \otimes 1 - 1 \otimes \psi_P^E\right)$. Let $\xi = \xi_{i_1}^{n_1} \xi_{i_2}^{n_2} \cdots \xi_{i_k}^{n_k} \in A_*$. We have $\Delta(\xi) = \prod_{t=1}^{k} \left(\Delta\left(\xi_{i_t}\right)\right)^{n_t}$. Let $p \in P$. According to Lemma

3.1, we have

$$\begin{aligned}
& \left(\Delta \otimes 1 - 1 \otimes \psi_P^E\right)(\xi \otimes p) \\
= & \prod_{t=1}^{k} \left(\Delta\left(\xi_{i_t}\right)\right)^{n_t} \otimes p - \xi \otimes 1 \otimes p \\
= & \prod_{t=1}^{k} \left(\sum_{0 \le j \le i_t} \xi_{i_t-j}^{2^j} \otimes \xi_j\right)^{n_t} \otimes p - \xi \otimes 1 \otimes p \\
= & \prod_{t=1}^{k} \left(\sum_{1 \le j \le i_t} \xi_{i_t-j}^{2^j} \otimes \xi_j\right)^{n_t} \otimes p \\
& + \sum_{s=1}^{k} \left(\xi_{i_s} \otimes 1\right)^{n_s} \left(\prod_{\substack{t=1 \\ t \neq s}}^{k} \left(\sum_{1 \le j \le i_t} \xi_{i_t-j}^{2^j} \otimes \xi_j\right)^{n_t}\right) \otimes p.
\end{aligned}$$

If $\xi \otimes p \in \ker\left(\Delta \otimes 1 - 1 \otimes \psi_P^E\right)$, then we claim that n_i is even for all i, that is $\ker\left(\Delta \otimes 1 - 1 \otimes \psi_P^E\right) \subseteq M \otimes_{\mathbb{Z}/2} P$. If not, there exists an i such that n_i is the largest odd number of all powers in ξ. Observing the above formulation, the term $\alpha \otimes \xi_{n_i} \otimes p$ can not be eliminated since it only occur once. So, ξ must belong to M. This proves that $A_* \square_E P = M \otimes_{\mathbb{Z}/2} P$, that is,

$$\mathrm{Ext}_{A_*}^{*,*}\left(\mathbb{Z}/2, A_* \otimes_{\mathbb{Z}/2} P\right) = \mathrm{Ext}_{A_*}^{*,*}\left(\mathbb{Z}/2, A_* \square_E P\right) \cong \mathrm{Ext}_E^{*,*}\left(\mathbb{Z}/2, P\right).$$

Since P is coaction trivial, we can easily conclude that

$$\mathrm{Ext}_E^{*,*}\left(\mathbb{Z}/2, P\right) \cong \mathrm{Ext}_E^{*,*}\left(\mathbb{Z}/2, \mathbb{Z}/2\right) \otimes_{\mathbb{Z}/2} P,$$

by computing the cobar complex of P over E directly. ■

Recall a well-known result.

Proposition 5.4 *$Ext_E^{*,*}\left(\mathbb{Z}/2, \mathbb{Z}/2\right) = \mathbb{Z}/2\left[\bar{\xi}_1, \bar{\xi}_2, \cdots\right]$, where* $\mathrm{bideg}\, \bar{\xi}_i = (1, 2^i - 1)$.

Proof. Consider the cobar complex of $\mathbb{Z}/2$ over E,

$$\mathbb{Z}/2 \longrightarrow \bar{E} \longrightarrow \bar{E} \otimes \bar{E} \longrightarrow \cdots,$$

where $\bar{E}$ is the argument algebra of E. The multiplication of this complex is the usual tensor product of graded module. So $Ext_E^{*,*}\left(\mathbb{Z}/2, \mathbb{Z}/2\right)$ must be a ring. Let $\Delta_E : E \longrightarrow E \otimes E$ be the coalgebra map. By Proposition 1.1, we get $\Delta_E(\xi_i) = 1 \otimes \xi_i + \xi_i \otimes 1$ for all i. Consider the i-th line, that is,

$$\overset{i-1}{\otimes} \bar{E} \xrightarrow{\Delta_E^{i-1}} \overset{i}{\otimes} \bar{E} \xrightarrow{\Delta_E^{i}} \overset{i+1}{\otimes} \bar{E}.$$

We claim that the cycle of the i-th line is $\overset{i}{\otimes}\xi_{n_i}$. It is clear that $\Delta_E^i\left(\overset{i}{\otimes}\xi_{n_i}\right) = 0$. Let $\xi \in \overset{i}{\otimes}\bar{E}$. If ξ is not of the form $\overset{i}{\otimes}\xi_{n_i}$, then we assume that $\xi = \overset{i}{\otimes}\alpha_i$, where $\alpha_i \in \bar{E}$ with a j such that $\alpha_j = \xi_{n_1}\xi_{n_2}\cdots\xi_{n_k}$, where $k > 1$. We have $\Delta_E^i(\alpha_j) = \prod_{t=1}^{k}\left(1 \otimes \xi_{n_t} + \xi_{n_t} \otimes 1\right) - 1 \otimes \alpha_j - \alpha_j \otimes 1$. Therefore, $\Delta_E^i(\xi) \neq 0$. We conclude that the cycle of the i-th line is $\overset{i}{\otimes}\xi_{n_i}$. To be continued. ∎

Since only $MU_{(2)}$ has a converging Adams spectral sequence, we replace MU by $MU_{(2)}$. By the properties of $MU_{(2)}$, we know that the calculations in above are the same. Therefore, we can get the same answers, that is, the E_2-term of $MU_{(2)}$ is $\mathbb{Z}/2\left[\bar{\xi}_1, \bar{\xi}_2, \cdots\right] \otimes_{\mathbb{Z}/2} P$. Consider the differentials of the Adams spectral sequence which converges to $\pi_*\left(MU_{(2)}\right)$,

$$d^r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r+1}.$$

Observing our E_2-term, since $\operatorname{bideg}\bar{\xi}_i = (1, 2^i - 1)$ and $\operatorname{bideg} b_j = (0, 2j)$ where $j \neq 2^l - 1$ for all l, we have $E_2^{s,t} = 0$ if $t - s$ is odd. It follows that all differentials vanish, that $d^r = 0$, because they shift degree $t - s$ by 1. Therefore, our Adams spectral sequence collapse, that is, $E_\infty^{*,*} \cong E_2^{*,*}$.

The last thing we need to do is to solve the group extension problem. Before we do this, we give a useful lemma first.

Lemma 5.5 *Let X be a space or a spectrum. The Adams spectral sequence with E_2-term equals to $\mathrm{Ext}_{A_*}^{*,*}(\mathbb{Z}/2, H_*(X;\mathbb{Z}/2))$ converges to $\pi_*\left(X_{(2)}\right)$. Let $x \in \pi_*\left(X_{(2)}\right)$ which is detected by $a \in E_\infty^{s,t}$. Then $2x$ is detected by $\xi_1 \otimes a \in E_\infty^{s+1,t+1}$.*

Here is the answer of this section.

Proposition 5.6 $\pi_*\left(MU_{(2)}\right) \cong \mathbb{Z}[m_1, m_2, \cdots]$ *where* $\deg m_i = 2i$.

Proof. We have the Adams spectral sequence E_∞-term, $Ext_E^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} P$ converging to $\pi_*\left(MU_{(2)}\right)$. If $d \neq 2^l - 1$ for all l, then let m_d be the element in $\pi_{2d}\left(MU_{(2)}\right)$, that is $m_d : S^{2d} \longrightarrow MU_{(2)}$, detected by b_d in $E_\infty^{0,2d}$. If $d = 2^l - 1$ for some l, let m_d be the element in $\pi_{2\left(2^l-1\right)}\left(MU_{(2)}\right)$, that is $m_d : S^{2\left(2^l-1\right)} \longrightarrow MU_{(2)}$, detected by $\bar{\xi}_d$ in $E_\infty^{1,2^d-1}$. Since $\left\{b_d \mid d \neq 2^l - 1 \text{ for all } l\right\} \cup \left\{\bar{\xi}_d \mid d = 2^l - 1 \text{ for some } l\right\}$ generate our E_∞-term, $\{m_d\}$ is the set of generators of $\pi_*\left(MU_{(2)}\right)$. Firstly, we claim that $m_i m_j \neq 0$ for all i, j. Since $m_i m_j$ is detected by an element α in $E_\infty^{0,*}$, $E_\infty^{1,*}$ or $E_\infty^{2,*}$ which all have no relations, it follows $m_i m_j \neq 0$. Otherwise, α is zero in the filtration quotient will become a relation. Secondly, by lemma 5.5, we know that m_i is torsion free for all i since our E_∞-term has no relation looks like $\xi_1 \otimes \_$. Thirdly, let $\sum_{i=1}^{k} n_i \alpha_i$ be in $\pi_*\left(MU_{(2)}\right)$. Assume $\sum_{i=1}^{k} n_i \alpha_i$ is in the j-th filtration, that is $\sum_{i=1}^{k} n_i \alpha_i \in F^j\left(\pi_*\left(MU_{(2)}\right)\right)$. Consider the natural projection

$$P : F^j \longrightarrow \frac{F^j}{F^{j+1}} \cong E_\infty^{j,*}.$$

If $P\left(\sum_{i=1}^{k} n_i \alpha_i\right) = 0$, then $P\left(\sum_{i=1}^{k} n_i \alpha_i\right)$ become a relation in $E_\infty^{*,*}$. It is a contradiction. It follows that $\sum_{i=1}^{k} n_i \alpha_i \neq 0$, that is $\pi_*\left(MU_{(2)}\right)$ has no relation. Therefore, we conclude that $\pi_*\left(MU_{(2)}\right) \cong \mathbb{Z}\left[m_1, m_2, \cdots\right]$ where $\deg m_i = 2i$. ■

6 Brown-Peterson Topological Splitting

Finally, we are on a good position to give a stable splitting of $MU_{(2)}$ which admits BP as a stable summand.

Proposition 6.1 (Brown-Peterson) *$MU_{(2)} \simeq \bigvee \sum^n BP$.*

Proof. As section 4, we have a stable map

$$f : BP \longrightarrow MU_{(2)}$$

which induces the inclusion map in $\mathbb{Z}/2$-homology, that is,

$$f_* : H_*\left(BP\right) \longrightarrow H_*\left(MU_{(2)}\right)$$

is the natural inclusion map. If $i \neq 2^l - 1$ for all l, let g_i be the map represents the generator m_i of $\pi_{2i}\left(MU_{(2)}\right)$ which is detected by $b_i \in E_\infty^{0,2i}$, i.e.,

$$g_i : S^{2i} \longrightarrow MU_{(2)}$$

and m_i is in the 0-th filtration. In the Adams tower,

$$\begin{tikzcd} & \vdots \arrow[d] & \\ & \overline{H(\mathbb{Z}/2)} \wedge MU_{(2)} \arrow[r] \arrow[d] & H(\mathbb{Z}/2) \wedge \overline{H(\mathbb{Z}/2)} \wedge MU_{(2)}\ , \\ S^n \arrow[r] & S^0 \wedge MU_{(2)} \arrow[r] & H(\mathbb{Z}/2) \wedge MU_{(2)} \arrow[ul] \end{tikzcd}$$

the bottom horizontal map, named by T, is the stable Hurewicz map from $\pi_*\left(MU_{(2)}\right)$ to $H_*\left(MU_{(2)}\right)$. Since m_i is in the 0-th filtration and not in the 1-st filtration, we have $T(g_i) \neq 0$. Therefore, $T(g_i)$ must be the generator of $H_{2i}\left(MU_{(2)}\right)$, i.e., b_i. Define

$$F : BP \wedge \left(\vee S^{2i}\right) \xrightarrow{f \wedge (\wedge g_i)} MU_{(2)} \wedge \left(\vee MU_{(2)}\right) \xrightarrow{h \circ \bar{h}} MU_{(2)},$$

where h is the ring map of ring spectrum $MU_{(2)}$ and $\bar{h}$ is the folding map. It follows that F_* is an isomorphism between $H_*(BP \wedge (\wedge S^{2i}))$ and $H_*\left(MU_{(2)}\right)$. By Hurewicz theorem and Whitehead theorem, we know that $BP \wedge (\wedge S^{2i}) \simeq MU_{(2)}$ stably, that is, $MU_{(2)} \simeq \bigvee \sum^n BP$. ■

References

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